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LETTER TO THE EDITOR

Generalized quantization scheme for central extensions of Lie algebras

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Abstract. We present the method for finding nonlinear Poisson–Lie group structures on the vector spaces and for their quantization. Explicit quantization formulas are proposed for arbitrary central extension of a Lie algebra.

Quantum algebras, or q-deformed algebras, have been useful in the investigation of many physical problems. As a matter of fact, the research in q-groups was indeed originated from physical problems. The interest in q-groups arose almost simultaneously in statistical mechanics and also in conformal field theories, in solid state physics and in the study of topologically non-trivial solutions of nonlinear equations. In this letter new examples related to inhomogeneous quantum groups of physical interest are developed and we give an explicit formula for their quantization (in the sense of [1]).

It is well known that the structure of a Poisson–Lie group on a vector space (considered as a commutative group by addition) is defined by an arbitrary Poisson bracket depending linearly on the coordinate functions, i.e. by a Lie algebra structure on an adjoint space. The quantization of such a Poisson bracket leads to the universal enveloping algebra of this Lie algebra [4], and the commutativity of the coproduct follows from the commutativity of a vector space. Therefore, we can attempt to deform the group structure on the vector space and in this way obtain the non-commutative and non-cocommutative Hopf algebras [1, 5].

We have the following model example.

Example.
Consider on R^3 the following Poisson bracket

$$\{H, x\} = x \qquad \{H, y\} = -y \qquad \{x, y\} = \frac{\sinh \gamma H}{\gamma} \qquad \gamma \neq 0 \text{ is a parameter.}$$

This bracket has the properties:

- The bracket is compatible with the following coproduct on $C^\infty(R^3)$
  $\Delta(x) = x \otimes \exp(\gamma H/2) + \exp(-\gamma H/2) \otimes x$,
  $\Delta(y) = y \otimes \exp(\gamma H/2) + \exp(-\gamma H/2) \otimes y$, $\Delta(H) = H \otimes 1 + 1 \otimes H$,
  which can be obtained from the group operation on R^3:

$$\begin{pmatrix} h \\ x \\ y \end{pmatrix} * \begin{pmatrix} h' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} h + h' \\ x \exp(\gamma h'/2) + \exp(-\gamma h/2) x' \\ y \exp(\gamma h'/2) + \exp(-\gamma h/2) y' \end{pmatrix}.$$

  Hence $(R^3, *)$ is the Poisson–Lie group [1, 3].

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- A simple substitution of this bracket by the commutator leads to the quantum $sl_q(2)$.

In this letter we study a method for finding such Poisson–Lie group structures on the vector spaces and their quantization.

A simple way to obtain a structure of non-commutative Lie group on a vector space is the consideration of a commutative extension of the commutative group. Let $\boldsymbol{H}$ and $\boldsymbol{V}$ be vector spaces, and ρ_1 and ρ_2 be two commuting representations of $\boldsymbol{H}$ (as the commutative Lie group) on the space $\boldsymbol{V}$. Then we can introduce the following Lie group $\tilde{G} = (\boldsymbol{H} \oplus \boldsymbol{V}, *)$, where

$$(h, v) * (h', v') = (h + h', \rho_1(h)v' + \rho_2(h')^{-1}v). \tag{1}$$

For this group the point (0,0) is the unit and

$$(h, v)^{-1} = (-h, -\rho_1(-h)\rho_2(h)v).$$

The coproduct Δ, counit ϵ and the antipode S on $C^\infty(\tilde{G})$ of the linear functions have the same form as the corresponding maps in [2]. In [2] (cf [6]) it was proposed to use such coproduct as a definition of the deformation of the Lie algebra structure $\Im$ on the space $(\boldsymbol{H} \oplus \boldsymbol{V})^*$. Let $\Im$ be the Lie algebra generated by H^i and X^m, where H^i form the basis of an Abelian subalgebra. Write down the deformation of the product in $U(\Im)$, which preserves (Δ, ϵ, S):

$$[x^r, x^s] = [x^r, x^s]_0 + \Phi^{rs}(H^k; \rho_1, \rho_2).$$

Here $[x^r, x^s]_0$ is the initial composition and the deforming functions $\Phi^{rs}(H^k; \rho_1, \rho_2)$ depend on ρ_1, ρ_2 and they are the power series of H^i. The question of the possibility of such deformation leads to the question of the possibility of defining a Poisson–Lie structure on the group $\tilde{G}$ for which the global Poisson bracket of the functions $\{h^i\}$ is zero and the Poisson bracket of the coordinate functions differs from the linear one only for functions of $\{h^i\}$. It will be good for the quantization of such a system, because we do not have a problem with ordering. Therefore we have the problem of finding a globally Poisson–Lie bracket on the group $\tilde{G}$ from the cocycle $\delta : g \to g \otimes g$, where g is the Lie algebra of $\tilde{G}$ and δ^* is our $[.,.]_0$ structure of the Lie algebra on the space $(\boldsymbol{H} \oplus \boldsymbol{V})^*$ with $[h^i, h^j] = 0$ for all i, j. Topologically, the group $\tilde{G}$ is a vector space and hence such Poisson bracket exists globally. A general analysis when such a bracket is linear + function of $\{h^i\}$ will be done in a forthcoming paper. In general we have

Lemma 1. If we have a cocycle δ for which $\boldsymbol{H}^*$ is an Abelian subalgebra, the global Poisson–Lie bracket of the functions $\{h^i\}$ on $\tilde{G}$ equals zero.

Now we are going to give an exposition of the method for finding the global Poisson bracket on $\tilde{G}$ from cocycle δ. After the change of variables $(h, v) \mapsto (h, \rho_2(h)v)$ we obtain the group structure on $\tilde{G}$ without ρ_2:

$$(h, v) \bullet (h', v') = (h + h', \rho(h)v' + v) \tag{2}$$

where $\rho(h) = \rho_1(h)\rho_2(h)$. Note that the differential of the change in the unit of $\tilde{G}$ is $\boldsymbol{Id}$, i.e. this change of variables does not change the Lie bialgebra structure on g. Further we work with $(\tilde{G}, \bullet)$. Let $h_\bullet$ be an infinitesimal version of ρ, i.e. $\rho(h) = \exp(h_\bullet)$.

Lemma 2. For the Lie algebra g of the group (2) we have $g = H \oplus V$, with $[h, h'] = 0$, $[v, v'] = 0$, $[h, v] = h_\bullet v$.

For the Lie group $\tilde{G}$ the constant vector field $(h', 0)$ is left-invariant for all $h' \in$H and the constant vector field $(0, v')$ is right-invariant for all $v \in$V. If $\pi^{ij}(h, v)$ is the Poisson–Lie bracket of the coordinate functions then from the theorem 1.2 in [3] we have

Lemma 3. For all k and l tensor $\partial\pi/\partial h^k$ is a left-invariant bivector field and $\partial\pi/\partial v^l$ is a right-invariant bivector field on the group $\tilde{G}$.

But $(\partial\pi^{ij}/\partial h^k, \partial\pi^{ij}/\partial v^l)|_{(0,0)}$ is our cocycle δ. From the Lemma 3 and the property $\pi(0,0) = 0$ we can find π globally.

Example.
H and V are one-dimensions, $h_1 \cdot v_1 = v_1$. Then $\tilde{G}$ is the Poisson–Lie group with bracket $\{h^1, v^1\} = \alpha v^1 + \beta(e^{h^1} - 1)$, where α and β are parameters. A simple substitution of this bracket by the commutator leads to the quantum Lie algebra $U_{\alpha,\beta}$ generated by h and v with

$$[h, v] = \alpha v + \beta(e^h - 1)$$

$$\Delta(h) = 1 \otimes h + h \otimes 1 \qquad \Delta(v) = e^h \otimes v + v \otimes 1$$

$$S(h) = -h,\ S(v) = e^{-h} v,\ S(1) = 1 \qquad \epsilon(h) = \epsilon(v) = 0 \qquad \epsilon(1) = 1.$$

Consider an arbitrary central extension of Lie algebra. Let $V = \langle v^i \rangle$ be a Lie algebra and $\langle h^1 \rangle \oplus V$ be its central-extension with cocycle Ω_0^{ij}, i.e.

$$[h^1, v] = 0 \qquad \forall v \in V \qquad [v^i, v^j] = \Omega_0^{ij} h^1 + C_k^{ij} v^k \tag{3}$$

where C_k^{ij} are the structuring constants of the Lie algebra V (here and below the summation convention over non-fixed repeated indices is in force). Let α and β be two commuting differentiations of V. Then $\tilde{G} = \langle h_1 \rangle \oplus V^*$ is the Poisson–Lie group with the product (1), where $\rho_1(h_1) = e^{\alpha^*}$, $\rho_2(h_1) = e^{\beta^*}$, and with the cocycle adjoint to (3). On $\tilde{G}$ coproduct, counit and antipode have the following form:

$$\begin{aligned}
&\Delta(h^1) = h^1 \otimes 1 + 1 \otimes h^1 \\
&\Delta(v^i) = e^{h^1 \otimes \alpha}(1 \otimes v^i) + e^{-\beta \otimes h^1}(v^i \otimes 1) \\
&S(h^1) = -h^1 \qquad S(v^i) = -\exp(-h^1\alpha)\exp(h^1\beta) v^i \\
&\epsilon(h^1) = \epsilon(v^i) = 0.
\end{aligned} \tag{4}$$

Theorem 1. The global Poisson–Lie bracket on $\tilde{G}$ with the cocycle adjoint to (3) is equal to

$$\{h^1, v^j\} = 0 \forall j \qquad \{v^i, v^j\} = \Omega^{ij}(h^1) + C_k^{ij} v^k \tag{5}$$

where

$$\left|\left|\Omega^{ij}(h^1)\right|\right| = \exp(-h^1\beta) \times \left(\int_0^{h^1} \exp(h'(\alpha + \beta)) \cdot \|\Omega_0^{ij}\| (\exp(h'(\alpha + \beta)))^t dh' \right) (\exp(-h^1\beta))^t.$$

Here t is the transposition of the matrix and $\left|\left|\Omega^{ij}\right|\right|$ is a matrix with the entries of Ω^{ij}.

Theorem 2. A simple substitution of this bracket (5) by the commutator leads to the quantization of the algebra (3), compatible with (4).

From these theorems we have a large class of quantum groups generated by h^1 and v^i with the relations (5) and (Δ, S, ϵ) from (4). Example 1 from [2] is a special case when V is a two-dimensional commutative Lie algebra.

In a forthcoming paper we shall describe in detail the case of the Virasoro algebra.

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